MATH 2850: HIGHER ORDER LINER ODEs AND VARIATION OF PARAMETERS (REVISITED)

DETERMINANTS OF 3×3 **MATRICES:** For a 3×3 matrix:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

EXAMPLE:

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{vmatrix} &= (3)\begin{vmatrix} -1 & 5 \\ 1 & 4 \end{vmatrix} - (1)\begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} + (2)\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) = -13 \end{aligned}$$

CRAMER'S RULE: The system of linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z &= c_1 \\ a_{21}x + a_{22}y + a_{23}z &= c_2 \\ a_{31}x + a_{32}y + a_{33}z &= c_3 \end{cases}$$

has a unique solution provided

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

In this case, the solution to the system can be found as follows: letting

$$D_1 = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$

EXAMPLE: Solve the following system using Cramer's Rule: $\begin{cases} x + y + z &= 3 \\ 2x - y + z &= 0 \\ -3x + 5y + 7z &= 7 \end{cases}$

Ans: $x = 1$, $y = 2$, $z = 0$

VARIATION OF PARAMETERS: (For third order linear ODEs):

Consider the ODE: $y''' + p(x)y'' + q(x)y' + r(x) = f(x)$ where p, q, r and f are continuous in an open interval.

Suppose $y_c = c_1y_1 + c_2y_2 + c_3y_3$. Define the following Wronskians:

$$W_{1,2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_{1,3} = \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad W_{2,3} = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}$$

and

$$W_{1,2,3} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Then $y_p = u_1y_1 + u_2y_2 + u_3y_3$ where

$$u_1' = \frac{f(x)W_{2,3}}{W_{1,2,3}}, \quad u_2' = -\frac{f(x)W_{1,3}}{W_{1,2,3}}, \quad u_3' = \frac{f(x)W_{1,2}}{W_{1,2,3}}$$

EXAMPLE: Consider $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2$.

- Find y_c.

  HINT: This is a Cauchy - Euler equation!

Ans: $y = c_1x^{-1/2} + c_2x^{1/2} + c_3x^2$

- Solve $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2$ using Variation of Parameters.

  HINT: Remember to put in the form $y''' + p(x)y'' + q(x)y' + r(x)y = f(x)$.

Ans: $y = c_1x^{-1/2} + c_2x^{1/2} + c_3x^2 + 2x^2\ln(x)$

HOMEWORK: Pg. 503: 1 - 31 every third odd.